5339: **Proposed by D.M. Bătinetu-Giurgiu**, "**Matei Basarab**" **National College**, **Bucharest**, **Romania and Neculai Stanciu** "**George Emil Palade**" **School**, **Buzău**, **Romania**

Calculate: $\int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}dx.$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Let $I := \int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}dx$ and $J := \int_0^{\pi/2} \frac{-3\sin x + 4\cos x}{3\cos x + 4\sin x}dx.$

Then $I + J = 8\int_0^{\pi/2} \frac{\cos x}{3\cos x + 4\sin x}dx = 8\int_0^{\pi/2} \frac{1}{3 + 4\tan x}dx =$

$[t := \tan x; dt = (1 + t^2)dx] = 8\int_0^{\infty} \frac{dt}{(3 + 4t)(1 + t^2)}.$

Since $\frac{1}{(3 + 4t)(1 + t^2)} = \frac{16}{25(4t + 3)} - \frac{4t - 3}{25(t^2 + 1)} = \frac{16}{25(4t + 3)} - \frac{4t}{25(t^2 + 1)} + \frac{3}{25(t^2 + 1)},$

$\int \frac{4dt}{4t + 3} = \ln(4t + 3) + c, \int \frac{2tdt}{t^2 + 1} = \ln(t^2 + 1) + c, \int \frac{dt}{t^2 + 1} = \tan^{-1}(t) + c$

then $\int \frac{dt}{(3 + 4t)(1 + t^2)} = \frac{2}{25}\ln\frac{(4t + 3)^2}{t^2 + 1} + \frac{3}{25}\tan^{-1}(t) + c.$

Noting that $\lim_{t\to\infty} \ln\frac{(4t + 3)^2}{t^2 + 1} = \ln 16 = 4\ln 2$ and $\lim_{t\to\infty} \tan^{-1}(t) = \frac{\pi}{2}$ we obtain

$I + J = \frac{8}{25}\left(2\ln\frac{(4t + 3)^2}{t^2 + 1} + 3\tan^{-1}(t)\right)_0^{\infty} = \frac{8}{25}\left(2(4\ln 2 - 2\ln 3) + 3 \cdot \frac{\pi}{2}\right) =$

$\frac{8}{25}\left(8\ln 2 - 4\ln 3 + \frac{3\pi}{2}\right).$

Since $J = \int_0^{\pi/2} \frac{d(3\cos x + 4\sin x)}{3\cos x + 4\sin x} = (\ln(3\cos x + 4\sin x))_0^{\pi/2} = \ln 4 - \ln 3$ then

$I = \frac{8}{25}\left(8\ln 2 - 4\ln 3 + \frac{3\pi}{2}\right) - (\ln 4 - \ln 3) = \frac{12\pi}{25} + \frac{14}{25}\ln 2 - \frac{7}{25}\ln 3.$

Remark.

Another way to evaluate integral $\int_0^{\pi/2} \frac{1}{3 + 4\tan x}dx$ consists in using of tangent of half argument, that is denoting $t := \tan\frac{x}{2}$ we obtain $dx = \frac{2dt}{t^2 + 1}$ and then

$\int_0^{\pi/2} \frac{1}{3 + 4\tan x}dx = 2\int_0^1 \frac{t^2 - 1}{(3t + 1)(t - 3)(t^2 + 1)}dt =$

$2\int_0^1 \left(\frac{2}{25(t - 3)} - \frac{4t - 3}{25(t^2 + 1)} + \frac{6}{25(3t + 1)}\right)dt =$

$2\left(\frac{2}{25}\int_0^1 \frac{dt}{t - 3} - \frac{2}{25}\int_0^1 \frac{(t^2 + 1)'}{t^2 + 1}dt + \frac{3}{25}\int_0^1 \frac{dt}{t^2 + 1} + \frac{2}{25}\int_0^1 \frac{dt}{3t + 1}\right) =:$

$2\left(\frac{2}{25}(\ln 2 - \ln 3) - \frac{2}{25}\ln 2 + \frac{3}{25} \cdot \frac{\pi}{4} + \frac{2}{25} \cdot \frac{2}{3}\ln 2\right) = \frac{3}{50}\pi + \frac{8}{75}\ln 2 - \frac{4}{25}\ln 3.$

Hence, $I + J = 8\left(\frac{3}{50}\pi + \frac{8}{75}\ln 2 - \frac{4}{25}\ln 3\right) = \frac{12}{25}\pi + \frac{64}{75}\ln 2 - \frac{32}{25}\ln 3$

and, therefore, $I = \left(\frac{12}{25}\pi + \frac{64}{25}\ln 2 - \frac{32}{25}\ln 3\right) - (2\ln 2 - \ln 3) = \frac{12}{25}\pi + \frac{14}{25}\ln 2 - \frac{7}{25}\ln 3.$